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$$
\begin{aligned}
\Gamma \text{ by } &\Gamma_0 + \Gamma_1 \cos(il+i'l'+i''\gamma) + \Gamma_2 \cos 2(il+i'l'+i''\gamma) + \dots, \\
l \text{ by } &l + l_1 \sin(il+i'l'+i''\gamma) + l_2 \sin 2(il+i'l'+i''\gamma) + \dots, \\
l' \text{ by } &l' + l'_1 \sin(il+i'l'+i''\gamma) + l'_2 \sin 2(il+i'l'+i''\gamma) + \dots, \\
\gamma \text{ by } &\gamma + \gamma_1 \sin(il+i'l'+i''\gamma) + \gamma_2 \sin 2(il+i'l'+i''\gamma) + \dots,
\end{aligned}
$$

and will have, for determining the new variables l, l', γ, a, a', e, precisely the same differential equations as we started with, provided we make all these substitutions in the function R, and regard the new variables L, L', Γ as connected with a, a', e by the relations

$$
\begin{aligned}
L &= L_0 + \tfrac{1}{2}(\theta_1 L_1 + 2\theta_2 L_2 + \dots), \\
L' &= L'_0 + \tfrac{1}{2}(\theta_1 L'_1 + 2\theta_2 L'_2 + \dots), \\
\Gamma &= \Gamma_0 + \tfrac{1}{2}(\theta_1 \Gamma_1 + 2\theta_2 \Gamma_2 + \dots).
\end{aligned}
$$

It will be perceived that as long as we are dealing with terms of R, whose arguments involve l or l' or both, the second members of the three equations, last written, have values which differ from the elliptic values of L, L' and Γ only by quantities of the second order with respect to disturbing forces. Hence, if we propose to neglect third order terms, until we have reduced R to a function of the argum't γ only, we can assume that L, L' and Γ which are the elements conjugate to the arguments l, l' and γ, are expressed throughout in terms of a, a' and e, in the same way as in the elliptic theory. It may be added that these third order terms are found in experience to be much smaller than those which arise in other ways.

[To be continued.]

SOME RELATIONS DEDUCED FROM EULER'S THEOREM ON THE CURVATURE OF SURFACES.

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LET α = angle of any normal section with line of maximum curvature,
k_a = curvature of this normal section,
k_m = maximum curvature,
k_n = minimum curvature; then by Euler's theorem :

$$k_a = k_m \cos^2\alpha + k_n \sin^2\alpha, \tag{1}$$

$$\therefore\ k_{a+\frac{1}{2}\pi} = k_m \sin^2\alpha + k_n \cos^2\alpha.$$

Adding, we obtain the well known relation :

$$k_a + k_{a+\frac{1}{2}\pi} = k_m + k_n = \text{constant}. \tag{2}$$

But multiplying we have

$$
\begin{aligned}
k_a k_{a+\frac{1}{2}\pi} &= (k_m^2 + k_n^2)\sin^2\alpha\cos^2\alpha + k_m k_n(\cos^4\alpha + \sin^4\alpha) \\
&= \tfrac{1}{4}(k_m + k_n)^2 \sin^2 2\alpha + k_m k_n \cos^2 2\alpha. \qquad (3)
\end{aligned}
$$

Place $$k_{m+n} = \tfrac{1}{2}(k_m + k_n) = \tfrac{1}{2}(k_a + k_{a+\frac{1}{2}\pi}) = k_{\frac{1}{4}\pi}, \tag{4}$$

$$k_{mn} = \sqrt{(k_m k_n)}, \tag{5}$$

then
$$k_\alpha k_{\alpha+\frac{1}{2}\pi} = k_{m+n}^2 \sin^2 2\alpha + k_{mn}^2 \cos^2 2\alpha. \qquad (3\ )$$

From (4) and (3′) we can form the quadratic:

$$k^2 - 2k_{m+n}k + k^2_{\ m+n}\sin^2 2\alpha + k_{mn}^2 \cos^2 2\alpha = 0, \tag{6}$$

whence
$$k_\alpha = k_{m+n} + \cos 2\alpha \sqrt{(k_{m+n}^2 - k_{mn}^2)}. \tag{7}$$

This formula although apparently more complicated than (1) is nevertheless of greater practical value for the following reasons:

The prin'l curvatures k_m and k_n on any surface are given by the quadratic

$$k^2 - \frac{(1+q^2)r - 2pqs + (1+p^2)t}{(1+p^2+q^2)^{\frac{3}{2}}}k + \frac{rt - s^{2*}}{1+p^2+q^2} = 0; \tag{8}$$

hence by (4)
$$k_{m+n} = \frac{(1+q^2)r - 2pqs + (1+p^2)t}{2(1+p^2+q^2)^{\frac{3}{2}}}, \tag{9}$$

and by (5)
$$k_{mn} = \frac{\sqrt{(rt - s^2)}}{1+p^2+q^2}. \tag{10}$$

Thus we see that these two quantities are directly given without solving a complete quadratic. Their geometrical signification is as follows:

We have evidently
$$k_{m+n} = k_{\frac{1}{4}\pi}, \tag{11}$$

also
$$\begin{aligned} k_{m+n} &= \int_0^\pi k_\alpha d\alpha \div \int_0^\pi d\alpha = \frac{1}{\pi}\int_0^\pi (k_m \cos^2\alpha + k_n \sin^2\alpha) d\alpha \qquad (12) \\ &= \frac{1}{\pi}\int_0^\pi [k_{m+n} + \cos 2\alpha \sqrt{(k_{m+n}{}^2 - k_{mn}{}^2)}] d\alpha \\ &= k_{m+n} + \left[\frac{\sqrt{(k_{m+n}{}^2 - k_{mn}{}^2)}}{2\pi}\sin 2\alpha\right]_0^\pi, \end{aligned}$$

it is therefore the mean curvature.

Since $k_m k_n$ is the measure of curvature according to Gauss, therefore $1 \div k_{mn}$ is the radius of a sphere of the same curvature (13); also

$$\begin{aligned} \frac{1}{k_{mn}} &= \int_0^\pi \frac{d\alpha}{k\alpha} \div \int_0^\pi d\alpha = \frac{1}{\pi}\int_0^\pi \frac{d\alpha}{k_m \cos^2\alpha + k_n \sin^2\alpha} \\ &= \frac{1}{\pi\sqrt{(k_m k_n)}}\int_0^\pi \frac{d\alpha \sqrt{(k_n \div k_m)}\sec^2\alpha}{1 + (k_n \div \mathrm{k}_m)\tan^2\alpha} \\ &= \frac{1}{\pi\sqrt{(\mathrm{k}_m \mathrm{k}_n)}}\left[\tan^{-1}\left\{\sqrt{\left(\frac{\mathrm{k}_n}{\mathrm{k}_m}\right)}\tan\alpha\right\}\right]_0^\pi \\ &= \frac{1}{\sqrt{(\mathrm{k}_m \mathrm{k}_n)}}. \end{aligned}$$

Thus we see $1 \div \mathrm{k}_{mn}$ = mean radius of curvature.

Replacing α by $\alpha + \frac{1}{4}\pi$ in (3′) we have

$$\mathrm{k}_{\alpha+\frac{1}{4}\pi}\mathrm{k}_{\alpha+\frac{3}{4}\pi} = \mathrm{k}_{m+n}^2 \cos^2 2\alpha + \mathrm{k}_{mn}^2 \sin^2 2\alpha.$$

Adding this to (3′) we obtain the very remarkable relation

$$\mathrm{k}_\alpha \mathrm{k}_{\alpha+\frac{1}{2}\pi} + \mathrm{k}_{\alpha+\frac{1}{4}\pi}\mathrm{k}_{\alpha+\frac{3}{4}\pi} = \mathrm{k}_{m+n}^2 + \mathrm{k}_{mn}^2 = \text{constant}. \tag{15}$$

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*This is Monge's notation; $p = (dr \div dx)$; $q = (dz \div dy)$; $r = (d^2z \div dx^2)$; $s = (d^2z \div dxdy)$; $t = (d^2z \div dy^2)$.

That is, the sum of the products of the curvatures of one pair of cross sections and of another pair of cross sections bisecting the first is *constant.*

This remarkable theorem, analogous to (2), is but a particular case of a more general one, which I give without proof, which is easy however.

If a pair of cross sections is continually bisected and after N bisections the product of curvatures taken, and if we add to this the product of the curvatures of a system of sections bisecting the first, the sum is constant.

From the examination of some special cases by Mr. Alex. S. Christie of U. S. Coast Surv. it seemed probable that in general the sum of N curvatures evenly distributed around a point; the sum of their products by two, by three . . . , by N—1 is also constant; but that their product depends on the azimuth. To prove this I proceed as follows:

We have by (7), if for brevity we place $\mathrm{k}_{m-n} = \sqrt{}(\mathrm{k}_{m+n}{}^2 - \mathrm{k}_{mn}{}^2)$, (16)

$$\mathrm{k}_{\alpha} = \mathrm{k}_{m+n} + \mathrm{k}_{m-n} \cos 2\alpha. \qquad (7')$$

We have then $\Sigma_0^{N-1}[\mathrm{k}_{a+(v\div N)\pi}] = \mathrm{N}\mathrm{k}_{m+n} = \text{constant},$ (17)

$$\begin{aligned}\Sigma_0^{N-1}[\mathrm{k}^2_{a+(v\div N)\pi}] &= \mathrm{N}\mathrm{k}^2_{m+n} + \mathrm{k}^2_{m-n}\Sigma_0^{N-1}\{\cos^2 2[\alpha+(\nu\div\mathrm{N})\pi]\} \\ &= \mathrm{N}\mathrm{k}^2_{m+n} + \tfrac{1}{2}\mathrm{k}^2_{m-n}\Sigma_0^{N-1}\{1+\cos 4[\alpha+(\nu\div\mathrm{N})\pi]\} \\ &= \mathrm{N}\mathrm{k}^2_{m+n} + \tfrac{1}{2}\mathrm{N}\mathrm{k}^2_{m-n} = \text{constant}. \qquad (18)\end{aligned}$$

Hence since $\Sigma_0^{N-1}[\mathrm{k}^2_{a+(v\div N)p}{}^{*}] = [\Sigma_0^{N-1}(\mathrm{k}_{a+(v\div N)p})]^2$

$$-2\Sigma_0^{N-1}[\mathrm{k}_{a+(v\div N)p}\mathrm{k}_{a+(v'\div N)p}],$$

we have the sum of the products by two

$$\begin{aligned}\Sigma_0^{N-1}[\mathrm{k}_{a+(v\div N)p}\mathrm{k}_{a+(v'\div N)p}] &= \tfrac{1}{2}\mathrm{N}^2\mathrm{k}^2_{m+n} - \tfrac{1}{2}\mathrm{N}\mathrm{k}^2_{m+n} - \tfrac{1}{4}\mathrm{N}(\mathrm{k}^2_{m+n} - \mathrm{k}^2_{mn} \\ &= \tfrac{1}{2}\mathrm{N}.\tfrac{1}{2}(2\mathrm{N}-3)\mathrm{k}^2_{m+n} + \tfrac{1}{4}\mathrm{N}\mathrm{k}^2_{mn} = \text{const.} \qquad (19)\end{aligned}$$

Again

$$\begin{aligned}\Sigma_0^{N-1}[\mathrm{k}^3_{a+(v\div N)p}] &= \mathrm{N}\mathrm{k}^3_{m+n} + \tfrac{3}{2}\mathrm{N}\mathrm{k}_{m+n}\mathrm{k}^2_{m-n} \\ &= \tfrac{5}{2}\mathrm{N}\mathrm{k}^3_{m+n} - \tfrac{3}{2}\mathrm{N}\mathrm{k}_{m+n}\mathrm{k}^2_{mn} = \text{constant}. \qquad (20)\end{aligned}$$

Since

$$\begin{aligned}\Sigma_0^{N-1}[\mathrm{k}^3_{a+(v\div N)p}] &= [\Sigma_0^{N-1}(\mathrm{k}_{a+(v\div N)p})]^3 - 3[\Sigma_0^{N-1}(\mathrm{k}_{a+(v\div N)p})\Sigma_0^{N-1}\ ) \\ &\quad (\mathrm{k}^2_{a+(v\div N)p}) - \Sigma_0^{N}(\mathrm{k}^3_{a+(v\div N)p})] - 6\Sigma_0^{N-1}(\mathrm{k}_{a+(v\div N)p}\mathrm{k}_{a+(v'\div N)p}\mathrm{k}_{a+(v''\div N)p} \\ &= -2[\Sigma_0^{N-1}(\mathrm{k}_{a+(v\div N)p})]^3 + 6\Sigma_0^{N-1}(\mathrm{k}_{a+(v\div N)p})\Sigma(\mathrm{k}_{a+(v\div N)p}\mathrm{k}_{a+(v'\div N)p}) \\ &\quad + 3\Sigma_0^{N-1}(\mathrm{k}_{a+(v\div N)p}) - 6\Sigma_0^{N-1}(\mathrm{k}_{a+(v\div N)p}\mathrm{k}_{a+(v'\div N)p}\mathrm{k}_{a+(v''\div N)p}), \qquad (21)\end{aligned}$$

we infer that the sum of the products by three of the N curvatures is const.

It is easy to see that the sum of the powers, as well as the products by four, five, etc., must be constant, but with this exception, that it does not hold for the sum of the Nth powers nor the product of the N curvatures; for we have

$$\begin{aligned}\Sigma_0^{N-1}[\mathrm{k}^N_{a+(v\div N)p}] &= \text{constant} + \mathrm{k}^N_{m-n}\Sigma_0^{N-1}\{\cos^N 2[\alpha+(\nu\div N)\pi]\} \\ &= \text{const.} + \frac{1}{2N\text{-}1}\mathrm{k}^{N}_{m-n}\,\Sigma_0^{N-1}\left\{\cos 2(N\alpha+\nu\pi)\right\} \\ &\qquad\qquad + \text{constant} \\ &= \text{const.} + \frac{N}{2N\text{-}1}\mathrm{k}^{N}_{m-n}\cos 2N\alpha; \qquad (22)\end{aligned}$$

therefore also the product of the curvatures is not constant.

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[*] For want of Greek sorts the subscript p is here written for π, and subsc'pt v stands for ν.